Preparatory excercises

Induction

We adopt the following axioms for addition and multiplication:

- $\forall x\ x + 0 = x$
- $\forall x, y\ x + S(y) = S(y + x)$
- $\forall x\ x \cdot 0 = 0$
- $\forall x, y\ x \cdot S(y) = (x \cdot y) + x$

Prove that:

1. $\forall x\ (S(x) = x + S(0))$
2. $\forall x\ (S(x) = S(0) + x)$
3. $\forall x\ (x + 0 = 0 + x)$
4. $\forall x\ (x = x \cdot S(0))$
5. $\forall x\ (x = S(0) \cdot x)$
6. $\forall x\ (x \cdot 0 = 0 \cdot x)$
7. $\forall x \forall y \forall z\ \big((x + y) + z = x + (y + z)\big)$
8. $\forall x \forall y \forall z\ \big(x \cdot (y + z) = (x \cdot y) + (x \cdot z)\big)$
9. $\forall x \forall y \forall z\ \big((x \cdot y) \cdot z = x \cdot (y \cdot z)\big)$
10. $\forall x \forall y\ (x + y = y + x)$
11. $\forall x \forall y\ (x \cdot y = y \cdot x)$
12. $\forall x \forall y \forall z\ \big((x + y) \cdot z = (x \cdot z) + (y \cdot z)\big)$

Algebra of sets

Prove that for every set A, B and C:

1. $A \setminus (A \cap B) = A \setminus B$
2. $(A \cup B) \setminus B = A \setminus B$

3. $(A \cap B) \cup B = B$
4. $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
5. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
6. $A \setminus (A \setminus B) = A \cap B$
7. $A \cap B = \emptyset \to A \setminus (B \cap C) = A$
8. $A \subseteq B \equiv (A \setminus B) = \emptyset$
9. $(A \setminus B) \cap B = \emptyset$

Relations, functions, orderings

1. Let Z be the set of all integers (positive and negative). Check whether the function $f : Z \to Z$ defined as $f(x) = x^2 - 1$ is:
   (a) injective,
   (b) surjective.
2. Is it true that:

   If R is a linear ordering on A and S is a linear ordering on B, then $R \cap S$ is a linear ordering on $A \cap B$?
3. Is it true that:

   If R is a linear ordering on A and S is a linear ordering on B, then $R \cup S$ is a linear ordering on $A \cup B$?
4. Assume that $(X, R) \cong (Y, S)$. Prove that:
   - If R is irreflexive on X, then S is irreflexive on Y,
   - If R is antisymmetric on X, then S is antisymmetric on Y,
   - If R is total on X, then S is total on Y,
   - If there are minimal elements in (X, R), then there are also minimal elements in (Y, S).
5. Let $X \sim Y$ be defined as $\exists f[f : X \xrightarrow[onto]{1-1} Y]$. Let $P(x)$ denote the power set of x. Prove that if $X \sim Y$, then $P(X) \sim P(Y)$.